ISRAEL JOURNAL OF MATHEMATICS, Vol. 67, No. 1, 1989

AXIOMS OF DETERMINACY AND BIORTHOGONAL SYSTEMS

BY

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ABSTRACT

If all Π^1_n games are determined, every non-norm-separable subspace X of $l^\infty(\mathbf{N})$ which is $w^* - \Sigma^1_{n+1}$ contains a biorthogonal system of cardinality $2^{\aleph_0}$. In Levy's model of Set Theory, the same is true of every non-norm-separable subspace of $l^\infty(\mathbf{N})$ which is definable from reals and ordinals. Under any of the above assumptions, X has a quotient space which does not linearly embed into $1^\infty(\mathbf{N})$.

1. Introduction

Let X be a Banach space. A biorthogonal system is a family $(x_\alpha, x^*_\alpha)_{\alpha\in I}$ of $X \times X^*$ such that the following conditions hold:

(i) $\sup \| x_\alpha \| \cdot \| x^*_\alpha \| < \infty$,

(ii) $x^*_\alpha(x_\beta) = 0$ if $\alpha \neq \beta$,

(iii) $x^*_\alpha(x_\alpha) = 1$.

In the present work, the set (x^*_α) will play no role and therefore we will call the family $(x_\alpha)_{\alpha\in I}$ itself a biorthogonal system.

It is immediate to check that the cardinality of a biorthogonal system in X cannot exceed the density character of X, and the question arises to know whether it is actually possible to construct in any Banach space X a biorthogonal system of cardinality dens(X). The answer is positive if X is separable; then a stronger result ([9]; see [7], p. 43) is actually available. However, if X is not separable, the answer is negative in general; a striking counterexample is the space $\mathscr{C}(K)$ constructed by K. Kunen with the continuum hypothesis (see

Received October 7, 1988

[11], pp. 1123–1129), which shows that it is not even true that uncountable biorthogonal systems can be constructed in any non-separable Banach space.

Still, positive results are available, and we will show in this note that non-separable subspaces of $l^\infty(\mathbf{N})$ which are not too pathological contain a biorthogonal system of cardinality $2^{\aleph_0}$. For instance, we will deduce from a suitable determinacy axiom that every nonseparable subspace X of $l^\infty(\mathbf{N})$ which belongs to the projective hierarchy (for the weak-* topology on $l^\infty(\mathbf{N})$) contains a biorthogonal system of cardinality $2^{\aleph_0}$.

The article consists of the juxtaposition of two techniques: In part 2.2, we use a game technique for constructing "big" perfect sets. Our reference for these techniques is Moschovakis' book ([10], Chapter 6). The other ingredient is Stegall's method [13], and its extension ([4], Lemme 4), which gives 2.3.

As a matter of notation, we use the modern notation (see [10]) for the classes of the projective hierarchy: analytic sets are Σ^1_1, coanalytic sets are Π^1_1, etc. $\mathrm{Det}(\Pi^1_n)$ means that all Π^1_n games on the integers are determined. $\mathrm{OD}(R)$ denotes the class of sets which can be defined in the language of set theory, with ordinals and real numbers as parameters. If R is a binary relation on a set P, we write interchangeably $(x, y) \in R$ and xRy. The w*-topology on $l^\infty(\mathbf{N})$ is the topology of pointwise convergence on its predual $l^1(\mathbf{N})$; observe that a subset of $l^\infty(\mathbf{N})$ is w*-Σ^1_n if and only if it is Σ^1_n as a subset of the Polish space $\mathbf{R}^{\mathbf{N}}$.

2. The main results

If Γ denotes a class of sets, we denote by $T(\Gamma)$ the following property:

> *Every subset A of $l^\infty(\mathbf{N})$ which is not separable in norm and belongs to the class Γ for the w*-topology contains a w*-perfect subset which is not separable in norm.*

The following proposition gathers several results about $T(\Gamma)$:

PROPOSITION 2.1. (1) *In ZFC, $T(\Sigma^1_1)$ holds.*

(2) *In ZFC, $T(\Sigma^1_2)$ is equivalent to $\forall \alpha\ \aleph_1^{L[\alpha]} < \aleph_1$, and to the perfect set theorem for coanalytic sets.*

(3) *$T(\mathrm{OD}(R))$ is equiconsistent with the existence of an inaccessible cardinal.*

(4) *In ZFC + $\mathrm{Det}(\Pi^1_n)$, $T(\Sigma^1_{n+1})$ holds.*

PROOF. We will first prove the assertions (1) and (4). They rely on the following general lemma.

LEMMA 2.2. *Let P be a Polish space, and R be a binary symmetric, reflexive* *and* Π_1^0 *relation on* P. *Assuming* $\mathrm{Det}(\Pi_n^1)$, *every* Σ_{n+1}^1 *subset* A *of* P *satisfies one of the following conditions*:

(i) *There is a countable subset* (a_n) *of* A *such that* $A \subseteq \bigcup_n \{y : a_n R y\}$.

(ii) *There is a perfect subset* K *of* A *such that for* $x \neq y$ *in* K, $(x, y) \notin R$.

PROOF OF LEMMA 2.2. Let (V_n) be a basis of P. The sets $F_n = \bar{V}_n$ will be called elementary closed sets. Let us first assume A is Π_n^1. Consider the following game $G(A)$ between two players I and II, played with the following rules: II starts the game by playing a pair (F_0^0, F_0^1) of elementary closed sets of diameter $\leqq 1$, such that for $x \in F_0^0$, $y \in F_1^0$, $(x, y) \notin R$ (if possible). I then chooses $\varepsilon(0) = 0$ or 1. II then chooses a pair (F_0^1, F_1^1) of elementary closed subsets of $F_{\varepsilon(0)}^0$, of diameter $\leqq 2^{-1}$, with for $x \in F_0^1$, $y \in F_1^1$, $(x, y) \notin R$, again if possible. I then chooses $\varepsilon(1) = 0$ or 1, and so on. We say that player II wins the run if (i) he has been able to play indefinitely, and (ii) if x is the unique element of $\bigcap_n F_{\varepsilon(n)}^n$, $x \in A$.

Clearly, this game can be viewed as a game on the integers, and its payoff is Π_n^1 (for II) if A is Π_n^1. So by our hypothesis, one of the players has a winning strategy.

Suppose first σ is a winning strategy for Player II, and define a function $f : \{0, 1\}^{\mathbf{N}} \to A$ by

$$\{f(\varepsilon)\} = \bigcap_n F_{\varepsilon(n)}^n .$$

It is clear that f is continuous and 1–1, so that $K = f\{0, 1\}^{\mathbf{N}}$ is a perfect subset of A. And by the rules of the game, if x and y are distinct points in K, one has $(x, y) \notin R$. So (ii) holds.

Suppose now I has a winning strategy σ. Say that a finite sequence of pairs of elementary closed sets s is x-admissible if s is a sequence which can be played by II in the game $G(A)$, I answering with his winning strategy σ, and moreover if $F(s)$ is the last closed set chosen by I (with $F(\varnothing) = P$), $x \in F(s)$. Now note that for each x in A, there must be an x-admissible sequence s which cannot be extended in an x-admissible sequence. Otherwise, player II would easily defeat I's strategy. Let us say that such a sequence is x-terminal. Now the set S of sequences which are x-terminal for some x in A is countable, so we can pick, for each s in S, a point $a(s)$ in A for which s is $a(s)$-terminal. We claim that every point of A is R-related to one of the $a(s)$'s. To see this, let $x \in A$, and let $s \in S$ be x-terminal. We show that $xRa(s)$. If not, we can find, as R is closed, two elementary closed sets F_0 and F_1, of small enough diameter, contained in

$F(s)$, with $F_0 \times F_1 \cap R = \varnothing$, and such that $x \in F_0$ and $a(s) \in F_1$. But then II can play (F_0, F_1) after s, and this extension must be admissible for one of x or $a(s)$. This contradiction proves our claim, and shows (i) holds.

It remains to study the case where A is Σ^1_{n+1}. Let then B be a Π^1_n subset of $\mathbf{N}^{\mathbf{N}} \times P$ with second projection A, and apply the preceding result to B and the closed relation S on $\mathbf{N}^{\mathbf{N}} \times P$ defined by $(\alpha, x)S(\beta, y)$ if xRy. If (i) holds for B with (α_n, a_n), (i) holds for A with (a_n). And if (ii) holds for B with a perfect set K, (ii) also holds for A with its projection. This concludes the proof of 2.2. □

We now come back to the proof of 2.1(1) and (4). The first assertion is a special case of the second one, since the determinacy of closed games is a theorem of ZFC. So we prove 2.1(4).

Let A be a Σ^1_{n+1} subset of $l^\infty(\mathbf{N})$ which is not norm-separable, and assume, with no loss of generality, that A is a subset of the unit ball P. For each $\varepsilon > 0$, the relation R_ε defined by

$$xR_\varepsilon y \leftrightarrow \| x - y \| \leqq \varepsilon$$

is closed in P and, by our hypothesis, there must be some ε for which property (i) of Lemma 2.2 does not hold for A and R_ε. By this lemma, it follows that there is a perfect subset K of A such that all points in K are at distance at least ε. This proves 2.1(4).

Let us now conclude the proof of 2.1. For (3), note that the existence of an inaccessible cardinal allows one to construct by forcing Levy's model M of ZFC ([6], [12]). And this model satisfies $T(\mathrm{OD}(R))$, by applying to the relations R_ε above, the following result of Louveau ([8], theorem 2.2): In M, if R is a closed relation and A in $\mathrm{OD}(R)$ is such that (i) if Lemma 2.2 does not hold for A, then A contains a Σ^1_1 subset for which (i) still does not hold. One can then apply 2.1(1). For the converse, one can use (2), as the statement $\forall \alpha\ \aleph_1^{L[\alpha]} < \aleph_1$ implies that $\aleph_1$ is inaccessible in L.

The implication $\forall \alpha\ \aleph_1^{L[\alpha]} < \aleph_1$ implies $T(\Sigma^1_2)$ can be obtained by a direct adaptation of the techniques of [8]. Let us finally observe that, conversely, $T(\Sigma^1_2)$ implies the perfect set theorem for Π^1_1 sets, because $\{0, 1\}^{\mathbf{N}}$ is canonically homeomorphic to a 1-separated subset of $l^\infty(\mathbf{N})$, hence any counterexample of the perfect set theorem for Π^1_1 sets in $\{0, 1\}^{\mathbf{N}}$ would yield a counterexample to $T(\Sigma^1_2)$. This concludes the proof of 2.1. □

We will now connect 2.1 with properties of non-separable subspaces of $l^\infty(\mathbf{N})$. Let us denote, for a class Γ, by $T^*(\Gamma)$ the following statement:

Every norm-closed subspace X of $l^\infty(\mathbf{N})$ which is not norm-separable and is in the class Γ for the w-topology contains a biorthogonal system of cardinality $2^{\aleph_0}$.*

Our next lemma is an easy consequence of ([4], lemma 4), which is itself an adaptation of a construction of Stegall [13].

LEMMA 2.3. *For every class Γ, $T(\Gamma)$ implies $T^*(\Gamma)$.*

PROOF. Let X be a norm-closed subspace of $l^\infty(\mathbf{N})$, not norm-separable, and in Γ. If $T(\Gamma)$ holds, X contains a w*-perfect subset K which is not norm-separable. Let $Y = \overline{\mathrm{sp}}(K)$ be the norm-closed linear span of K. Y is a subspace of X, and one easily checks from its definition that Y is Σ^1_1 (in fact $F_{\sigma\delta}$) for the w*-topology; it is therefore representable in the terminology of [4], and not norm-separable since it contains K. Now by ([4], lemma 4), Y, and hence X, contains a w*-perfect subset which is also a biorthogonal system, obviously of cardinality c.

Putting together 2.1 and 2.3 gives our main result:

THEOREM 2.4. *Let X be a norm-closed and not norm-separable subspace of $l^\infty(\mathbf{N})$. Under any of the following conditions, X contains a biorthogonal system of cardinality c:*

(1) *Assuming $\forall \alpha\ \aleph_1^{L[\alpha]} < \aleph_1$, if X is w*-Σ^1_2.*

(2) *In Levy's model, if X is definable from reals and ordinals.*

(3) *Assuming* $\mathrm{Det}(\Pi^1_n)$, *if X is w*-Σ^1_{n+1}.*

Let us note that the statement $T^*(\Sigma^1_1)$ is the main result of [4]; however the techniques of [4] do not give the stronger statement $T(\Sigma^1_1)$. Let us emphasize that statement 2.4(2) means that in Levy's model any explicit subspace of $l^\infty(\mathbf{N})$, in a precise and very general meaning of the word, is separable or contains a biorthogonal system of cardinality c.

Our techniques lead to further investigation of the "reasonable subspaces" of $l^\infty(\mathbf{N})$. For instance, one has:

PROPOSITION 2.5. *Let X be a non-norm-separable subspace of $l^\infty(\mathbf{N})$ which satisfies one of the assumptions of 2.4. Then X contains a closed subspace Y which is not a countable intersection of closed hyperplanes.*

PROOF. Let us observe that 2.1 and 2.3 actually show that under the assumptions of 2.4, the space X contains a subspace Z which is w*-analytic and not norm-separable. Now [4] shows that either Z contains $l^1(c)$, or that (Z_1^*, w^*) is an angelic compact space.

If Z contains $l^1(c)$, so does X; hence $l^\infty(\mathbf{N})$ is a quotient of X, and *a fortiori* $l^\infty(\mathbf{N})/c_0(\mathbf{N})$ is a quotient of X. Let $Q: X \to l^\infty(\mathbf{N})/c_0(\mathbf{N})$ be a quotient map, and $Y = \operatorname{Ker} Q$. Since $l^\infty(\mathbf{N})/c_0(\mathbf{N})$ does not linearly embed in $l^\infty(\mathbf{N})$, it is easily seen that Y is not the intersection of countably many closed hyperplanes.

If (Z_1^*, w^*) is angelic, let $(x_\alpha)_{\alpha\in c}$ be a biorthogonal system in Z, and (x_α^*) the corresponding subset of Z^*. Let $Y = \bigcap_\alpha \ker x_\alpha^*$. We claim Z/Y does not embed in $l^\infty(\mathbf{N})$. For otherwise, the space $Y^\perp = \overline{\operatorname{sp}}^*(x_\alpha^*)$ would be w*-separable. But by angelicity, every $y^* \in Y^\perp$ is the w*-limit of a sequence in $\operatorname{sp}(x_\alpha^*)$; and this easily implies that for every countable subset (y_n^*) of $Y^\perp$, there is an α such that $y_n^*(x_\alpha) = 0$ for all n, and hence $Y^\perp$ cannot be w*-separable.

In both cases, Z contains a closed subspace Y which is not the countable intersection of closed hyperplanes in Z, hence neither in X. □

REMARKS AND EXAMPLES 2.6. (1) Recall that a biorthogonal system (x_α) in a Banach space X is called a Markushevich basis (see [9]) if it satisfies:

(i) $\overline{\operatorname{sp}}^{\|\ \|}(x_\alpha) = X$,

(ii) $\bigcap_\alpha \ker x_\alpha^* = \{0\}$.

Every separable Banach space has a Markushevich basis [9]. The proof of 2.5 actually shows the following: If a non-separable Banach space is such that w*-dens$(X^*) = \aleph_0$, and (X_1^*, w^*) is an angelic compact space, then X has no Markushevich basis (see [14] for a stronger result). Since these properties are hereditary, X does not even contain uncountable Markushevich basic families. Let us emphasize two consequences:

(a) If Y is a separable Banach space and if Y^* contains a non-separable subspace Z which has a Markushevich basis, then Y contains $l^1(\mathbf{N})$. The special case $Z = l^1(c)$ is classical; and conversely, it is clear that $l^1(c) \subset Y^*$ if $l^1(\mathbf{N}) \subset Y$.

(b) By [1] and the above, if Y is separable and does not contain $l^1(\mathbf{N})$, and Z is a dual with the R.N.P. which is isomorphic to a subspace of Y^*, then Z is separable. Note that Z is not assumed to be w*-closed in Y^*.

(2) Using C.H., K. Kunen [5] (see [11], Theorem 7.7) has constructed a scattered separable non-metrizable compact space K, such that $X = \mathscr{C}(K)$ satisfies the following property: If F is any subset of X of cardinality $\aleph_1$, there is a point x in F with $x \in \overline{\operatorname{conv}}^{\|\ \|}(F \setminus \{x\})$. In particular, X contains no uncountable biorthogonal system. Observe that X is isometric to a subspace Y of $l^\infty(\mathbf{N})$, since K is separable; but the proof of 2.3 shows that X contains no w*-compact

non-norm-separable subset, and thus the space Y is necessarily very irregular for the w*-topology. Also [2], Theorem 3.3, shows that even 2.5 fails for X, i.e. every closed subspace of X is a countable intersection of closed hyperplanes, and X has "few" subspaces. It would be nice to know if X also has "few" operators, as suggested in ([11], p. 1129).

(3) It would be interesting to drop the assumption "X is a subspace of $l^\infty(\mathbf{N})$" in 2.4, to obtain larger classes of spaces in which non-separability implies the existence of uncountable biorthogonal systems; for instance, by [13] and [1], this is so if X is a dual space. However, note that the space $V = \mathscr{C}(\omega_1)$ is such that every subspace or quotient of it which is isomorphic to a subspace of $l^\infty(\mathbf{N})$ is already separable; hence different techniques seem to be needed for extending our results.

Acknowledgements

This work was completed while the first-named author was visiting the University of Missouri — Columbia during the Academic Year 1987–88, and the second-named author was visiting U.M.C. in January 1988. The authors are pleased to thank the Department of Mathematics of U.M.C., and in particular N. J. Kalton, for their warm hospitality and support.

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